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Optimal growth under discounting in the two-sector Robinson–Solow–Srinivasan model: a dynamic programming approach†

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On the Occasion of the 60th Birthday of Kazuo Nishimura

We use a version of a two-sector model to provide a strong form of a "folk-theorem" on the existence of a threshold discount factor such that: (i) for discount factors above this threshold value, optimal behavior is qualitatively similar to that in the corresponding undiscounted optimization problem, (ii) for discount factors below this threshold value, optimal behavior is qualitatively different from that in the undiscounted case. In the process, we provide an explicit solution of a non-linear optimal policy function for all discount factors above the threshold value. Our bifurcation analysis is conducted by using the dynamic programming approach, and we exploit the convex structure of our model to develop a variation of the standard method in dynamic programming used to identify the optimal policy correspondence.

Keywords: Dynamic programming; Value function; Optimal policy correspondence; Bifurcation analysis; Threshold discount factor

JEL Classification: C61; D90; O41

1. Introduction

It is a "folk theorem" of the literature on optimal economic growth that there is a *threshold discount factor* such that the stability properties of optimal programs are qualitatively the same as those obtained for the undiscounted case for all discount factors above that threshold, and that the qualitative behavior changes for all discount factors below that threshold||. In this paper, we provide a particularly strong version of this result in the context of the two-sector version of the Robinson–Solow–Srinivasan model, referred to henceforth as the RSS model#.

Specifically we show that there is a threshold discount factor $\rho^* \equiv (1/\xi)$, (where ξ represents, in the context of the model, the rate of transformation between machines today

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†This essay is dedicated to Kazuo Nishimura, dear friend and admired colleague, on the occasion of his sixtieth birthday. We are grateful to the *Center for Analytic Economics* at Cornell University and to the *Center for a Livable Future* at Johns Hopkins University for research support. We thank a referee for useful comments on an earlier version of this paper.

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||This is admittedly a rather loose statement, but that is the nature of "folk theorems". For more precise versions, see especially Refs. [3,7,14].

#See Refs. [12,15,17]. A continuous-time version of the general RSS model, under discounting, was analyzed by Stiglitz [18]. An analysis of the discrete-time version of the general RSS model, when future utilities are not discounted, is contained in Ref. [4].

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and machines tomorrow, while maintaining full-employment of both labor and capital), such that (i) for all discount factors $\rho \in (\rho^*, 1)$, the optimal policy correspondence is the *same* as the optimal policy function for the corresponding undiscounted optimization problem†; and (ii) for all discount factors $\rho \in (0, \rho^*)$, the optimal policy correspondence is *not the same* as the optimal policy function for the corresponding undiscounted optimization problem‡. Thus, ρ^* is a bifurcation value of the discount factor, marking a qualitative change in optimal behavior as one crosses this threshold.

The explicit solution of an optimal policy function allows one to study optimal transition dynamics, as well as optimal long-run behavior. Ours is one of the few examples of an explicit solution of the optimal policy correspondence as a *non-linear* optimal policy function, arising from a dynamic optimization problem in economics¶. As in the undiscounted case, two features of the transition dynamics (for $\rho \in (\rho^*, 1)$) are worth noting. First, starting from initial stocks below the modified golden-rule, optimality always requires an over-building phase, leading to stocks above the modified golden-rule stock, even when the long-run behavior warrants a convergence to the modified golden-rule stock. Second, even when it is feasible to fully utilize the available capital stock, it is not always optimal to do so, making it possible to observe excess-capacity and production of machines in the same time period.

Our earlier analysis of the corresponding undiscounted optimization model was accomplished by a synthesis of the dynamic programming and the value-loss approaches to characterizing optimality. In contrast, in this paper, we rely exclusively on the dynamic programming approach. In this regard, it is worthwhile to make two brief comments of a technical nature. First, exploiting the convex structure of our model, we use a variation of the standard method in dynamic programming to solve for the value function and the optimal policy correspondence, and this requires us to specify a "candidate" value function and a "candidate" policy correspondence on a relatively "small" subset of the state space. This method might be useful in the application of the dynamic programming approach in other contexts, since convex structures are often an integral part of many dynamic models in economics. Second, the key to our bifurcation analysis is the study of a quadratic inequality, and the bifurcation value of the discount factor is seen to emerge naturally as the smaller root associated with the corresponding quadratic equation.

2. Preliminaries

2.1 The two-sector version of the RSS model

A single consumption good is produced by infinitely divisible labor and machines with the further Leontief specification that a unit of labor and a unit of a machine produce a unit of the consumption good. In the investment-goods sector, only labor is required to produce

†See Ref. [5] for the complete analysis of the corresponding undiscounted optimization problem in the two-sector RSS model, and an explicit solution of the optimal policy function in that case.

‡In particular, the first part of this statement implies that we obtain an explicit solution for the optimal policy correspondence when $\rho \in (\rho^*, 1)$. For the second part of this statement, it is not important to actually obtain an explicit solution of the optimal policy correspondence for $\rho \in (0, \rho^*)$ and we do not. See, however, our concluding remarks in the final section of the paper for some hints about what it might look like.

¶The best known instance of an explicit solution of a non-linear optimal policy function is the Weitzman example, reported in Ref. [13], and discussed extensively in Refs. [1,7,9].

machines, with $a > 0$ units of labor producing a single machine. Machines depreciate at the rate $0 < d < 1$. A constant amount of labor, normalized to unity, is available in each time period $t \in \mathbb{N}$, where $\mathbb{N}$ is the set of non-negative integers. Thus, in the canonical formulation surveyed in Ref. [8], the collection of production plans (x,x'), the amount x' of machines in the next period (tomorrow) from the amount x available in the current period (today), is given by the *transition possibility set:*

$$\Omega = \{(x, x') \in \mathbb{R}^2_+ : x' - (1-d)x \geq 0, \quad \text{and} \quad a(x' - (1-d)x) \leq 1\}$$

where $z \equiv (x' - (1-d)x)$ is the number of machines that are produced, and $z \geq 0$ and $az \leq 1$ respectively formalize constraints on reversiblity of investment and the use of labor. Associated with Ω is the transition correspondence, $\Gamma : \mathbb{R}_+ \to \mathbb{R}_+$, given by $\Gamma(x) = \{x' \in \mathbb{R}_+ : (x, x') \in \Omega\}$. For any $(x, x') \in \Omega$, one can consider the amount y of the machines available for the production of the consumption good, leading to a correspondence $\Lambda : \Omega \to \mathbb{R}_+$ with

$$\Lambda(x, x') = \{y \in \mathbb{R}_+ : 0 \leq y \leq x \quad \text{and} \quad y \leq 1 - a(x' - (1-d)x)\}$$

Welfare is derived only from the consumption good and is represented by a linear function, normalized so that y units of the consumption good yields a welfare level y.A *reduced form utility function*, $u : \Omega \to \mathbb{R}_+$ with $u(x, x') = \max\{y \in \Lambda(x, x')\}$ indicates the maximum welfare level that can be obtained today, if one starts with x of machines today, and ends up with x' of machines tomorrow, where $(x, x') \in \Omega$. Intertemporal preferences are represented by the present value of the stream of welfare levels, using a discount factor $\rho \in (0, 1)$.

An *economy E* consists of a triple (a,d,ρ), and the following concepts apply to it. A *program* from x_o is a sequence $\{x(t), y(t)\}$ such that $x(0) = x_o$, and for all $t \in \mathbb{N}$, $(x(t), x(t+1)) \in \Omega$ and $y(t) = \max \Lambda((x(t), x(t+1))$. A *program* $\{x(t), y(t)\}$ is simply a program from $x(0)$, and associated with it is a *gross investment sequence* $\{z(t+1)\}$, defined by $z(t+1) = (x(t+1) - (1-d)x(t))$ for all $t \in \mathbb{N}$. It is easy to check that every program $\{x(t), y(t)\}$ is bounded by $\max\{x(0), 1/ad\} \equiv M(x(0))$, and in particular every program $\{x(t), y(t)\}$ from $x \in Z \equiv [0, (1/ad)]$ is bounded by $(1/ad)$.

For every program $\{x(t), y(t)\}$, we have $\sum_{t=0}^{\infty} \rho^t u(x(t), x(t+1)) < \infty$. A program $\{\bar{x}(t), \bar{y}(t)\}$ from x_o is called *optimal* if:

$$\sum_{t=0}^{\infty} \rho^t u(x(t), x(t+1)) \leq \sum_{t=0}^{\infty} \rho^t u(\bar{x}(t), \bar{x}(t+1))$$

for every program $\{x(t), y(t)\}$ from x_o. A program $\{x(t), y(t)\}$ is called *stationary* if for all $t \in \mathbb{N}$, we have $(x(t), y(t)) = (x(t+1), y(t+1))$. A *stationary optimal program* is a program that is stationary and optimal; in this case, the stationary level of $x(t)$ is called a *stationary optimal stock*.

The parameter, $\xi = (1/a) - (1-d)$, which figures prominently in our analysis, represents the rate of transformation between machines today and machines tomorrow, while maintaining full-employment of both labor and capital. Without further explicit mention, ξ will be assumed to be positive in what follows.

2.2 The modified golden rule

A *modified golden rule* is a pair $(\hat{x}, \hat{p}) \in \mathbb{R}^2_+$ such that $(\hat{x}, \hat{x}) \in \Omega$ and:

$$u(\hat{x}, \hat{x}) + (\rho - 1)\hat{p}\hat{x} \geq u(x, x') + \hat{p}(\rho x' - x) \quad \text{for all } (x, x') \in \Omega. \tag{1}$$

Given a modified golden-rule $(\hat{x}, \hat{p}) \in \mathbb{R}^2_+$, we know that $\hat{x}$ is a stationary optimal stock (see, for example, [8]).

Our first proposition establishes the existence of a modified golden-rule. A distinctive feature of the RSS model with discounting is that we can describe the modified golden-rule stock explicitly in terms of the parameters of the model.

PROPOSITION 1. *There is $(\hat{x}, \hat{p}) \in \mathbb{R}^2_+$ such that $(\hat{x}, \hat{x}) \in \Omega$, where $\hat{x}$ is independent of ρ, and:*

$$u(\hat{x}, \hat{x}) + (\rho - 1)\hat{p}\hat{x} \geq u(x, x') + \hat{p}(\rho x' - x) \quad \textit{for all } (x, x') \in \Omega. \tag{1}$$

Proof. We define:

$$\hat{x} = 1/(1 + ad), \quad \text{and} \quad \hat{p} = 1/(1 + \rho\xi)$$

where $\xi = (1/a) - (1 - d)$. Clearly, $(\hat{x}, \hat{p}) \in \mathbb{R}^2_+$ and $(\hat{x}, \hat{x}) \in \Omega$, $\hat{x} \in \Lambda(\hat{x}, \hat{x})$. Further, $\hat{x}$ is independent of ρ. We will show that $(\hat{x}, \hat{p})$ is a modified golden-rule of the economy (Ω, u, ρ).

Let $(x, x') \in \Omega$, and let $y \in \Lambda(x, x')$. Define $\beta(x, x', y) = [1 - ax' + (1 - d)ax] - y$, and $\alpha(x, x', y) = x - y$. Note that:

$$y \leq [1 - ax' + (1 - d)ax] \tag{2}$$

so that $\beta(x, x', y) \geq 0$, and $y \leq x$, so that $\alpha(x, x', y) \geq 0$.

Define $x'' \geq x'$, such that:

$$y = [1 - ax'' + (1 - d)ax] \tag{3}$$

Then $(x, x'') \in \Omega$, and $y \in \Lambda(x, x'')$. This allows us to deal with a full-employment point, and permits us to use the equality in equation (3) rather than the inequality in equation (2). Then, we can compute:

$$\begin{aligned} y + \rho\hat{p}x'' - \hat{p}x &= [1 - ax'' + (1 - d)ax] + \rho\hat{p}x'' - \hat{p}x \\ &= 1 + (\rho\hat{p} - a)x'' + [(1 - d)a - \hat{p}]x \end{aligned} \tag{4}$$

Using the fact that $y \equiv x - \alpha(x, x', y)$, we have:

$$[1 - ax'' + (1 - d)ax] = x - \alpha(x, x', y)$$

and this can be rewritten conveniently as:

$$x'' = (1/a) + [(1 - d) - (1/a)]x + [\alpha(x, x', y)/a] = (1/a) - \xi x + [\alpha(x, x', y)/a] \tag{5}$$

We can now use equation (5) in equation (4) to get:

$$
\begin{aligned}
y+\rho\hat{p}x''-\hat{p}x &= 1+(\rho\hat{p}-a)x''+[(1-d)a-\hat{p}]x \\
&= 1+(\rho\hat{p}-a)(1/a)+[-(\rho\hat{p}-a)\xi+(1-d)a-\hat{p}]x+(\rho\hat{p}-a)[\alpha(x,x',y)/a] \\
&= (\rho\hat{p}/a)+[-(\rho\hat{p}-a)\xi+(1-d)a-\hat{p}]x+(\rho\hat{p}-a)[\alpha(x,x',y)/a] \qquad (6)
\end{aligned}
$$

Now, by definition of $\hat{p}$, we have:

$$
\begin{aligned}
[-(\rho\hat{p}-a)\xi+(1-d)a-\hat{p}] &= -(\rho\xi+1)\hat{p}+(1-d)a+ab\xi \\
&= -1+(1-d)a-[(1-d)-(1/a)]a = 0 \qquad (7)
\end{aligned}
$$

So, using equation (7) in equation (6), we obtain:

$$y+\rho\hat{p}x''-\hat{p}x = (\rho\hat{p}/a)+(\rho\hat{p}-a)[\alpha(x,x',y)/a] \qquad (8)$$

Finally, note that by definition of $\beta(x,x',y)$, we have $\beta(x,x',y)=a(x''-x')$, so that $x''=x'+[\beta(x,x',y)/a]$, and using this in equation (8), we get:

$$y+\rho\hat{p}x'-\hat{p}x = (\rho\hat{p}/a)-(a-\rho\hat{p})[\alpha(x,x',y)/a]-\rho\hat{p}[\beta(x,x',y)/a] \qquad (9)$$

For $(x,x')=(\hat{x},\hat{x})$ and $y=\hat{y}=\hat{x}$, it is easy to check that $\alpha(x,x',y)=0$ and $\beta(x,x',y)=0$, so that:

$$\hat{y}+\rho\hat{p}\hat{x}-\hat{p}\hat{x} = (\rho\hat{p}/a) \qquad (10)$$

Combining equation (9) and (10), and noting that $a>\rho\hat{p}$, we get:

$$y+\rho\hat{p}x'-\hat{p}x \leq \hat{y}+\rho\hat{p}\hat{x}-\hat{p}\hat{x} \qquad (11)$$

which establishes equation (1). □

2.3 The dynamic programming approach

In this subsection, we describe the dynamic programming approach to characterizing optimality§. Underlying the approach are (a) a value function, and (b) a policy correspondence. Connecting these two objects of interest is the functional equation of dynamic programming.

Using standard methods, one can establish that there exists an optimal program from every $x \in X \equiv [0,\infty)$. Thus, we can define a *value function*, $V: X \to \mathbb{R}$ by:

$$V(x) = \sum_{t=0}^{\infty} \rho^t u(\bar{x}(t),\bar{x}(t+1)) \qquad (12)$$

where $\{\bar{x}(t),\bar{y}(t)\}$ is an optimal program from x. Then, it is straightforward to check that V is concave, non-decreasing and continuous on X. Further, it can be verified that V is, in fact, increasing on X.

It can be shown that for each $x \in X$, the Bellman equation:

$$V(x) = \max_{x' \in \Gamma(x)} \{u(x,x')+\rho V(x')\} \qquad (13)$$

§Our exposition is deliberately brief. For a comprehensive account of the dynamic programming approach to optimal growth models, see Ref. [6].

holds. For each $x \in X$, we denote by $h(x)$ the set of $x' \in \Gamma(x)$ which maximize $\{u(x, x') + \delta V(x')\}$ among all $x' \in \Gamma(x)$. That is, for each $x \in X$,

$$h(x) = \arg\left[\max_{x' \in \Gamma(x)} \{u(x, x') + \rho V(x')\}\right] \tag{14}$$

Then, a program $\{x(t), y(t)\}$ from $x \in X$ is an optimal program from x if and only if it satisfies the equation: $V(x(t)) = u(x(t), x(t+1)) + \delta V(x(t+1))$ for $t \geq 0$; that is, if and only if $x(t+1) \in h(x(t))$ for $t \geq 0$. We call h the (optimal) *policy correspondence*.

It is easy to verify, using $\rho \in (0, 1)$, that the function V, defined by equation (12), is the *unique* continuous function on $Z \equiv [0, (1/ad)]$ which satisfies the functional equation of dynamic programming, given by equation (13).

The connection between the value function in the dynamic programming approach and the modified golden-rule may be noted as follows.

Proposition 1 shows that:

$$(\hat{x}, \hat{p}) = (1/(1 + ad), 1/(1 + \rho\xi)) \tag{15}$$

is a modified golden-rule of the economy (Ω, u, ρ). A standard argument can now be used to show that $\hat{x}$ is a stationary optimal stock of the economy (Ω, u, ρ). Consequently, we have:

$$V(\hat{x}) = \hat{x}/(1 - \rho) \tag{16}$$

It is easy to verify that:

$$V(x) - \hat{p}x \leq V(\hat{x}) - \hat{p}\hat{x} \quad \text{for all } x \geq 0 \tag{17}$$

Choosing $x = \hat{x} + \varepsilon$ (with $\varepsilon > 0$) in (16), and letting $\varepsilon \to 0$, we obtain:

$$V'_{+}(\hat{x}) \leq \hat{p} \tag{18}$$

Using equation (18) and (1), we have:

$$V'_{+}(\hat{x}) \leq \hat{p} = 1/(1 + \rho\xi) < (a/\rho) \tag{19}$$

3. Basic properties of the optimal policy correspondence

In this section, we describe basic properties of the optimal policy correspondence, with minimal restrictions on the parameters of our model. These preliminary properties will help us to establish the principal bifurcation result in the next section.

To this end, we describe three regions of the state space:

$$A = [0, \hat{x}], \quad B = (\hat{x}, k), \quad C = [k, \infty) \tag{20}$$

where $k = \hat{x}/(1 - d)$. In addition, we define a function, $g : X \to X$, by:

$$g(x) = \begin{cases} (1 - d)x & \text{for } x \in C \\ \hat{x} & \text{for } x \in B \\ (1/a) - \xi x & \text{for } x \in A \end{cases} \tag{21}$$

We refer to g as the "pan map", in view of the fact that its graph resembles a pan.

We further subdivide the region B into two regions as follows:

$$D = (\hat{x}, 1), \quad E = [1, k) \tag{22}$$

and define a correspondence, $G : X \rightarrow X$, by:

$$G(x) = \begin{cases} \{(1-d)x\} & \text{for } x \in C \\ [(1-d)x, \hat{x}] & \text{for } x \in E \\ [(1/a) - \xi x, \hat{x}] & \text{for } x \in D \\ \{(1/a) - \xi x\} & \text{for } x \in A \end{cases} \tag{23}$$

The main result of this section can be summarized in the following proposition.

PROPOSITION 2. *The optimal policy correspondence, h, satisfies:*

$$h(x) \subset \begin{cases} \{g(x)\} & \textit{for all } x \in A \cup C \\ G(x) & \textit{for all } x \in B \end{cases}$$

The proposition is established in the following subsections by proving three lemmas, dealing with the behavior of the policy correspondence in the regions *C, A* and *B* (in that order).

3.1 The policy function for high initial stocks

We first examine the policy function for high values of the initial stock (region C). Denoting (as in Section 3) the policy correspondence by h, we now claim that for all $x \in C$, it is given precisely by g.

LEMMA 1. *The optimal policy correspondence, h, satisfies:*

$$h(x) = \{g(x)\} \;\; \textit{for} \;\; x \in C \tag{24}$$

Proof. Let $x \in C$, and suppose contrary to equation (24), we have $z \in h(x)$, such that $z \neq g(x)$. Since the irreversible investment constraint implies that $z \geq (1-d)x$, there is $\varepsilon > 0$ such that $z = g(x) + \varepsilon$. Then, we have the following:

$$V(x) = u(x, g(x) + \varepsilon) + \rho V(g(x) + \varepsilon), \quad V(x) \geq u(x, g(x)) + \rho V(g(x) \tag{25}$$

This yields:

$$\begin{aligned} u(x, g(x)) - u(x, g(x) + \varepsilon) &\leq \rho[V(g(x) + \varepsilon) - V(g(x)] \leq \rho V'_{+}(g(x))\varepsilon \\ &\leq \rho V'_{+}(\hat{x})\varepsilon \leq [\rho/(1 + \rho\xi)]\varepsilon \end{aligned} \tag{26}$$

the first inequality following from equation (25), the second from concavity of V, the third from concavity of V and the fact that $g(x) \geq \hat{x}$ (from equation (21)), and the last from equation (19). Now, note that: $[\rho/(1 + \rho\xi)] \leq [1/(1 + \xi)] = a/(1 + ad) < a$, so that we have:

$$u(x, g(x)) - u(x, g(x) + \varepsilon) < a\varepsilon \tag{27}$$

by using equation (26).

On the other hand:

$$u(x, g(x) + \varepsilon) \leq 1 - a[g(x) + \varepsilon - (1 - d)x] = 1 - a\varepsilon$$

and:

$$u(x, g(x)) = \min\{x, 1\} = 1$$

so that:

$$u(x, g(x)) - u(x, g(x) + \varepsilon) \geq a\varepsilon$$

which contradicts equation (27) and establishes the lemma. □

3.2 The policy function for low initial stocks

We next examine the policy correspondence for low values of the initial stock (region A), and establish that for all $x \in A$, it is also given precisely by g.

LEMMA 2. *The optimal policy correspondence, h, satisfies:*

$$h(x) = \{g(x)\} \text{ for all } x \in A \tag{28}$$

Proof. Suppose, contrary to equation (28), there is $x \in [0, \hat{x}]$, and $z \in h(x)$, such that $z \neq g(x)$. We consider two cases: (i) $z < g(x)$, (ii) $z > g(x)$.

Case (*i*). : Suppose $z < g(x)$. Note that $u(x, z) \leq x$. Thus, by the optimality principle, we have:

$$V(x) = u(x, z) + \rho V(z) < x + \rho V(g(x)) \tag{29}$$

the inequality following from the fact that V is increasing. But, using equation (21), we have:

$$1 - a[g(x) - (1 - d)x] = 1 - [1 - a\xi x] + a(1 - d)x = x \tag{30}$$

so that:

$$u(x, g(x)) = \min\{x, 1 - a[g(x) - (1 - d)x]\} = x \tag{31}$$

by using equation (30). Thus, we must have:

$$V(x) \geq u(x, g(x)) + \rho V(g(x)) = x + \rho V(g(x))$$

which contradicts equation (29).

Case. (*ii*): Suppose $z > g(x)$. Let $\varepsilon = z - g(x)$, so that $\varepsilon > 0$. Note that:

$$u(x, z) \leq 1 - a[z - (1 - d)x] = x - a\varepsilon \tag{32}$$

the equality in equation (32) following from equation (21), as in equation (30) above. Using the optimality principle,

$$\begin{aligned} V(x) &= u(x,z) + \rho V(z) \leq x - a\varepsilon + \rho[V(z) - V(g(x))] + \rho V(g(x)) \\ &\leq x - a\varepsilon + \rho V'_{+}(g(x))\varepsilon + \rho V(g(x)) \leq x - a\varepsilon + \rho V'_{+}(\hat{x})\varepsilon + \rho V(g(x)) \\ &\leq x - a\varepsilon + [\rho/(1+\rho\xi)]\varepsilon + \rho V(g(x)) < x + \rho V(g(x)) \end{aligned} \tag{33}$$

the first inequality following from equation (29), the second inequality following from concavity of V, the third inequality following from concavity of V and the fact that $g(x) \geq \hat{x}$ by equation (21), the fourth and fifth inequalities following from equation (19). But, since $(x, g(x)) \in \Omega$ and $u(x, g(x)) = x$ (as in equation (31) above) we must have $V(x) \geq x + \rho V(g(x))$, which contradicts equation (33), proving the lemma. □

3.3 The policy correspondence for the middle section

We now turn to the "middle section" (B). In this subsection, we demonstrate the basic property that the optimal policy correspondence for this region is included in the correspondence G, leaving a more precise specification to the next section.

LEMMA 3. *The optimal policy correspondence, h, satisfies:*

$$h(x) \subset G(x) \quad \textit{for all } x \in B \tag{34}$$

Proof. Consider, first, the initial stocks in the range $E = [1, k)$. We claim that:

$$h(x) \subset G(x) \ \text{ for } x \in E \tag{35}$$

Suppose, contrary to equation (35), there is $x \in E$, and $z \in h(x)$, such that z does not belong to $G(x)$. Since investment is irreversible, $z \geq (1-d)x$. Thus, by equation (23), we must have $z > \hat{x}$; denote $(z - \hat{x})$ by ε, so that $\varepsilon > 0$.

Note that $1 - a[\hat{x} - (1-d)x] \leq 1 \leq x$, so that:

$$u(x, \hat{x}) = 1 - a[\hat{x} - (1-d)x]$$

Also, we have:

$$u(x,z) \leq 1 - a[z - (1-d)x] = u(x,\hat{x}) - a\varepsilon \tag{36}$$

Using the optimality principle, we have:

$$\begin{aligned} V(x) &= u(x,z) + \rho V(z) \leq u(x,\hat{x}) - a\varepsilon + \rho V(z) \\ &= u(x,\hat{x}) - a\varepsilon + \rho[V(z) - V(\hat{x})] + \rho V(\hat{x}) \ \ \leq \rho V'_{+}(\hat{x})\varepsilon - a\varepsilon + V(x) < V(x) \end{aligned} \tag{37}$$

the first inequality in equation (37) following from equation (36), the second from concavity of V and the optimality principle, and the last from equation (19). The contradiction in equation (37) establishes our claim.

Consider next initial stocks in the range $D = (\hat{x}, 1)$. We claim now that:

$$h(x) \subset G(x) \ \text{ for } \ x \in D \tag{38}$$

Suppose, contrary to (38), that there is $x \in D$, and $z \in h(x)$, such that z does not belong to $G(x)$. Denote $[(1/a) - \xi x]$ by $H(x)$. There are two cases to consider: (i) $z < H(x)$, (ii) $z > H(x)$.

Case (i). : Suppose $z < H(x)$. Note that $u(x, z) \leq x$. Thus, by the optimality principle, we have:

$$V(x) = u(x, z) + \rho V(z) < x + \rho V(H(x)) \tag{39}$$

the inequality following from the fact that V is increasing. But, by definition of H:

$$1 - a[H(x) - (1 - d)x] = 1 - [1 - a\xi x] + a(1 - d)x = x$$

so that:

$$u(x, H(x)) = \min\{x, 1 - a[H(x) - (1 - d)x]\} = x$$

and we must have:

$$V(x) \geq u(x, H(x)) + \rho V(H(x)) = x + \rho V(H(x))$$

which contradicts equation (39).

Case (ii). : Suppose $z > H(x)$. Since z is not in $G(x)$, we must have $z > \hat{x}$. Let $\varepsilon = z - \hat{x}$, so that $\varepsilon > 0$. Note that:

$$1 - a[\hat{x} - (1 - d)x] \leq 1 - a[(1/a) - \xi x] + a(1 - d)x = x$$

so that:

$$u(x, \hat{x}) = \min\{1 - a[\hat{x} - (1 - d)x], x\} = 1 - a[\hat{x} - (1 - d)x] \tag{40}$$

Also,

$$u(x, z) \leq 1 - a[z - (1 - d)x] = u(x, \hat{x}) - a\varepsilon \tag{41}$$

the equality in equation (41) following from equation (39). Using the optimality principle,

$$\begin{aligned} V(x) &= u(x, z) + \rho V(z) \leq u(x, \hat{x}) - a\varepsilon + \rho[V(z) - V(\hat{x})] + \rho V(\hat{x}) \\ &\leq V(x) - a\varepsilon + \rho V'_{+}(\hat{x})\varepsilon < V(x) \end{aligned} \tag{42}$$

the first inequality in equation (42) following from equation (41), the second from concavity of V and the optimality principle, and the last from equation (19). The contradiction in equation (42) establishes our claim equation (38) and hence the lemma. □

3.4 A characterization of the pan policy map

Given Proposition 2, it is clear that the optimal policy correspondence is fully determined, except for the middle section, B. If the modified golden-rule stock $\hat{x}$ belongs to $h(x)$ for some $x \in B$, then it does so for every $x \in B$: this is the basic content of the characterization result which we now state and prove.

LEMMA 4. *The map g satisfies:*

$$g(x) \in h(x) \text{ for every } x \in B$$

if and only if there is some $\bar{x} \in B$ *such that* $g(\bar{x}) \in h(\bar{x})$.

Proof. The "only if" part of the statement being obvious, we proceed to establish the "if" part. So, assume that there is some $\bar{x} \in B$ such that $\hat{x} \in h(\bar{x})$.

Now let x be an arbitrary point in $B' = [\hat{x}, k]$. Then, $\hat{x} = (1-d)k \geq (1-d)x$, and:

$$a[\hat{x} - (1-d)x] \leq a[\hat{x} - (1-d)\hat{x}] = ad\hat{x} < 1$$

so that $(x, \hat{x}) \in \Omega$. Define $y = 1 - a[\hat{x} - (1-d)x]$. Then, we have:

$$x[1 - a(1-d)] \geq \hat{x}[1 - a(1-d)] = 1 - a[\hat{x} - (1-d)\hat{x}] - a(1-d)\hat{x} = 1 - a\hat{x}$$

so that $y \leq x$. Thus, $y \in \Lambda(x, \hat{x})$, and we have:

$$\begin{aligned} V(x) &\geq y + \rho V(\hat{x}) = 1 - a[\hat{x} - (1-d)x] + \rho V(\hat{x}) \\ &= 1 - a[\hat{x} - (1-d)\hat{x}] + a(1-d)(x - \hat{x}) + \rho V(\hat{x}) = V(\hat{x}) + a(1-d)(x - \hat{x}) \end{aligned} \quad (43)$$

Applying equation (43) to $x = \bar{x}$, and noting that $\hat{x} \in h(\bar{x})$, we have:

$$V(\bar{x}) = V(\hat{x}) + a(1-d)(\bar{x} - \hat{x}) \quad (44)$$

Similarly, applying equation (43) to $x = k$, and noting (from Lemma 2) that $\hat{x} \in h(k)$, we have:

$$V(k) = V(\hat{x}) + a(1-d)(k - \hat{x}) \quad (45)$$

Since $\bar{x} \in B$, there is $\lambda \in (0, 1)$, such that $\bar{x} = \lambda\hat{x} + (1-\lambda)k$. Using this in equation (44) and (45), we get:

$$V(\lambda\hat{x} + (1-\lambda)k) = \lambda V(\hat{x}) + (1-\lambda)V(k)$$

Since V is concave on B', this implies (see Ref. [11], exercise A(3)) that V is affine on B' and using equation (45):

$$V(x) = [V(\hat{x}) - a(1-d)\hat{x}] + a(1-d)x \text{ for all } x \in B'$$

In view of equation (43) then, we must have $V(x) = y + \rho V(\hat{x})$ for every $x \in B$, so that $\hat{x} \in h(x)$ for every $x \in B$. □

4. The bifurcation result

In this section, we state and prove the principal bifurcation result of this paper. We show that when $\rho\xi > 1$, then $g(x) \in h(x)$ for every $x \in X$, so the "pan policy" is optimal. And, when $\rho\xi < 1$, then for every $x \in B$, we have $g(x) \notin h(x)$. Thus, there is a qualitative change in behavior of optimal programs when the unit value of $\rho\xi$ is crossed, making $\rho\xi = 1$ a bifurcation point.

THEOREM 1

(i) *If* $\rho\xi > 1$, *the optimal policy correspondence, h, satisfies:*

$$g(x) \in h(x) \ \text{for all}\, x \in X$$

(ii) *If* $\rho\xi < 1$, *the optimal policy correspondence, h, satisfies:*

$$g(x) \notin h(x) \ \text{for all}\, x \in X$$

Our bifurcation analysis of optimal behavior with respect to changes in the discount factor may be seen as a continuation of the line of research reported in Refs. [2,9,10]. However, the fact that we present an explicit solution of the optimal policy correspondence (for $\rho\xi > 1$) makes our result sharper, albeit in the context of a more specific two-sector model‖.

We will prove this result in the following subsections. It turns out that the proof of both parts of the result depend crucially on a quadratic inequality. We establish this technical result before turning to the proof of the theorem.

4.1 On a quadratic inequality

In the analysis of the next two subsections, we will need to compare two magnitudes, (a/ρ) and $[1 - \rho\xi(1 - a\xi)]$. The connection between the two magnitudes leads to the study of a quadratic inequality, and the bifurcation value (of the theorem above) then emerges from the smaller root of the associated quadratic equation.

LEMMA 5. *The inequality:*

$$(a/\rho) \leq 1 - \rho\xi(1 - a\xi) \tag{46}$$

holds for $\rho \in (0, 1)$ *if and only if:*

$$\rho\xi \geq 1$$

Proof. We can write equation (46) as:

$$\rho^2\xi(1 - a\xi) - \rho + a \leq 0$$

In order to study this inequality, let us define, for $\rho \in \mathbb{R}$,

$$F(\rho) = \rho^2\xi(1 - a\xi) - \rho + a \tag{47}$$

‖On the other hand, explicit solutions of non-linear optimal policy functions (such as the tent map) have been obtained, given a discount factor, by *choosing* the transition possibility set and the utility function appropriately (dependent on the given discount factor). Such constructs, though useful in understanding other issues, are ill-suited to conducting bifurcation analysis with respect to the discount factor, given the transition possibility set and the utility function. For a full discussion of this point, and for the relevant literature, the reader is referred to Mitra and Nishimura [9].

Consider the equation: $F(\rho) = 0$. This has two roots; call them ρ_1 and ρ_2. One root is clearly $\rho_1 = (1/\xi)$, as one can check by substituting this value in equation (47). Also, since:

$$\rho_1\rho_2 = a/\xi(1 - a\xi)$$

and $(1 - a\xi) = a(1 - d)$, the other root is:

$$\rho_2 = a/(1 - a\xi) = a/[a(1 - d)] = 1/(1 - d) > 1$$

Clearly, $F(\rho) \leq 0$ holds if and only if $\rho \in [\rho_1, \rho_2]$. Since $\rho \in (0, 1)$, equation (46) can hold if and only if $\rho \geq \rho_1 = (1/\xi)$. This establishes the result. □

4.2 The non-optimality of the pan policy

We start by establishing the second part of Theorem 1, since it involves a direct application of the dynamic programming approach, after obtaining a restriction on the slope of the value function at the modified-golden rule stock.

LEMMA 6. *Suppose $\hat{x} \in h(x)$ for some $x \in (\hat{x}, k)$. Then, we must have:*

$$V'_{-}(\hat{x}) \geq (a/\rho) \tag{48}$$

Proof. Let $x \in (\hat{x}, k)$ be given such that $\hat{x} \in h(x)$ Then, there is $\varepsilon > 0$, such that for all $z \in I \equiv (\hat{x} - \varepsilon, \hat{x} + \varepsilon)$, we have $(x, z) \in \Omega$ and $\{1 - a[z - (1 - d)x]\} < x$, so that:

$$u(x, z) = 1 - a[z - (1 - d)x]$$

Define $F(x) = \{z : (x, z) \in \Omega\}$, and for $z \in F(x)$, define:

$$W(z) = u(x, z) + \rho V(z)$$

For $z \in I$, we have:

$$W(z) = 1 - az + a(1 - d)x + \rho V(z)$$

Since $\hat{x} \in I$, we obtain:

$$W'_{-}(\hat{x}) = -a + \rho V'_{-}(\hat{x}) \tag{49}$$

For $z \in I$, with $z < \hat{x}$, we must have:

$$W(z) = u(x, z) + \rho V(z) \leq V(x) = W(\hat{x})$$

the second equality following from the fact that $\hat{x} \in h(x)$. Thus, we have the first-order necessary condition: $W'_{-}(\hat{x}) \geq 0$. Using this in equation (49), we get equation (48).

PROPOSITION 3. *Suppose the parameters of the model satisfy the condition:*

$$\rho\xi < 1 \tag{50}$$

Then, for every $x \in (\hat{x}, k)$, we have:

$$\hat{x} \notin h(x)$$

Proof. Suppose, contrary to the Proposition, there is some $x \in B$ such that $\hat{x} \in h(x)$. Pick $z < \hat{x}$ such that $h(z) \in B$; certainly for all $z < \hat{x}$, with z close to $\hat{x}$, this is true by continuity of the policy function on $[0, \hat{x}]$. By the optimality principle, we obtain:

$$V(\hat{x}) - V(z) = (\hat{x} - z) + \rho[V(h(\hat{x})) - V(h(z))] \tag{51}$$

Also, we have:

$$\frac{V(\hat{x}/(1-d)) - V(\hat{x})}{[\hat{x}/(1-d)] - \hat{x}} = a(1-d) \tag{52}$$

So, we can write:

$$\begin{aligned} V(h(z)) - V(h(\hat{x})) &= \left\{\frac{V(h(z)) - V(h(\hat{x}))}{h(z) - h(\hat{x})}\right\}[h(z) - h(\hat{x})] \\ &= \left\{\frac{V(h(z)) - V(\hat{x})}{h(z) - \hat{x}}\right\}[\xi(\hat{x} - z)] \\ &\geq \left\{\frac{V(\hat{x}/(1-d)) - V(\hat{x})}{[\hat{x}/(1-d)] - \hat{x}}\right\}[\xi(\hat{x} - z)] = a(1-d)[\xi(\hat{x} - z)] \end{aligned} \tag{53}$$

the inequality in equation (53) following from the concavity of V, and the last equality following from equation (52). Using equation (53) in equation (51), we obtain:

$$V(\hat{x}) - V(z) = (\hat{x} - z) - \rho[V(h(z)) - V(h(\hat{x}))] \leq (\hat{x} - z) - \rho\xi a(1-d)(\hat{x} - z)$$

and:

$$\left\{\frac{V(\hat{x}) - V(z)}{\hat{x} - z}\right\} \leq 1 - \rho\xi a(1-d) \tag{54}$$

By Lemma 6, we have:

$$V'_{-}(\hat{x}) \geq (a/\rho) \tag{55}$$

Combining equation (54) and (55), and the concavity of V, we obtain:

$$(a/\rho) \leq V'_{-}(\hat{x}) \leq \left\{\frac{V(\hat{x}) - V(z)}{\hat{x} - z}\right\} \leq 1 - \rho\xi a(1-d) \tag{56}$$

Noting that $a(1-d) = (1 - a\xi)$, we obtain:

$$(a/\rho) \leq 1 - \rho\xi(1 - a\xi) \tag{57}$$

Using Lemma 5, equation (57) can hold only for $\rho \geq (1/\xi)$. This contradicts condition equation (50), and establishes the result. □

4.3 The optimality of the pan policy

We now turn to the first part of the result, stated in Theorem 1. A standard method of proving optimality of a policy is to propose a "candidate" value function and verify that it satisfies the functional equation of dynamic programming. By the uniqueness of the solution to the functional equation, this identifies the candidate value function as the actual value function, and therefore, identifies the associated candidate policy correspondence as the actual policy correspondence#.

#For applications of this standard method in solving for value functions, see Ref. [16], and the references cited there.

Our approach is a variation on the above method. Note that the above method does not rely on any convex structures. However, our model has a convex structure and this leads to the simplification that it is enough to propose a "candidate" value function on a relatively "small" subset of the entire state space. We feel that this technique might be of wider use than in the specific context of our two-sector model.

Pick $0 < K < \hat{x}$ such that $g(K) < k \equiv \hat{x}/(1-d)$. Define $Y = [K, k]$, $\bar{B} = [\hat{x}, k]$, $Y' = [K, \hat{x})$, and define a function $W : Y \to \mathbb{R}$ by:

$$\left.\begin{aligned} &W(\hat{x}) = \hat{x}/(1-\rho) \\ &W(x) = W(\hat{x}) + a(1-d)(x-\hat{x}) && \text{for } x \in Y, x > \hat{x} \\ &W(x) = W(\hat{x}) - [1 - \rho a\xi(1-d)](\hat{x}-x) && \text{for } x \in Y, x < \hat{x} \end{aligned}\right\} \tag{58}$$

and a function $f : Y \to \mathbb{R}$ by:

$$\left.\begin{aligned} &f(x) = \hat{x} && \text{for } x \in \bar{B} \\ &f(x) = g(x) && \text{for } x \in Y' \end{aligned}\right\} \tag{59}$$

Finally, denote the restriction of the transition correspondence Γ to Y by A:

$$A(x) = \Gamma(x) \cap Y \quad \text{for all } x \in Y$$

LEMMA 7. *If $\rho\xi > 1$, then the function $W : Y \to \mathbb{R}$, defined by equation* (58)*, satisfies the following functional equation of dynamic programming:*

$$W(x) = \max_{x' \in A(x)} \{u(x, x') + \rho W(x')\} \tag{60}$$

for all $x \in Y$, and:

$$f(x) = \arg\max_{x' \in A(x)} \{u(x, x') + \rho W(x')\} \tag{61}$$

for all $x \in Y$.

Proof. We consider two cases: (i) $x \in \bar{B}$, (ii) $x \in Y'$, and analyze each in turn.

Case (*i*). In this case, $(x, \hat{x}) \in \Omega$ and it can be checked that $y \equiv 1 - a[\hat{x} - (1-d)x]$ is in $\Lambda(x, \hat{x})$, so that $u(x, \hat{x}) = 1 - a[\hat{x} - (1-d)x]$. Thus, we have:

$$\begin{aligned} u(x, \hat{x}) + \rho W(\hat{x}) &= 1 - a[\hat{x} - (1-d)x] + \rho W(\hat{x}) = a(1-d)(x-\hat{x}) + \hat{x} + \rho W(\hat{x}) \\ &= a(1-d)(x-\hat{x}) + W(\hat{x}) = W(x) \end{aligned}$$

For $(x, x') \in \Omega$ with $x' \in Y$ and $x' > \hat{x}$, we have:

$$\begin{aligned} u(x, x') + \rho W(x') &\leq 1 - a[x' - (1-d)x] + \rho W(x') \\ &= 1 - a[\hat{x} - (1-d)x] + a(\hat{x} - x') + \rho[W(\hat{x}) + a(1-d)(x' - \hat{x})] \\ &= W(x) + a(x' - \hat{x})[\rho(1-d) - 1] < W(x) \end{aligned}$$

For $(x, x') \in \Omega$ with $x' \in Y$ and $x' < \hat{x}$, we have:

$$u(x, x') + \rho W(x') \leq 1 - a[x' - (1 - d)x] + \rho W(x')$$
$$= 1 - a[\hat{x} - (1 - d)x] + a(\hat{x} - x')\rho[W(\hat{x}) - (1 - \rho a\xi(1 - d)(\hat{x} - x')] \tag{62}$$
$$= W(x) + (\hat{x} - x')[a - \rho(1 - \rho a\xi(1 - d))]$$

Since $1 - \rho a\xi(1 - d) = 1 - \rho\xi(1 - a\xi)$, we can use Lemma 5 to conclude that $[a - \rho(1 - \rho a\xi(1 - d))] < 0$, since $\rho\xi > 1$. Thus, the expression on the last line of equation (62) is less than $W(x)$, and so $u(x, x') + \rho W(x') < W(x)$. This completes our verification that equation (60) and (61) hold for every $x \in \bar{B}$.

Case (ii). In this case, $(x, g(x)) \in \Omega$, and $y \equiv [1 - a(g(x) - (1 - d)x)]$ is in $\Lambda(x, g(x))$, so that $u(x, g(x)) = y = [1 - a(g(x) - (1 - d)x)]$. Thus, we have:

$$\begin{aligned} u(x, g(x)) + \rho W(g(x)) &= [1 - a(g(x) - (1 - d)x)] + \rho W(g(x)) = [1 - a(\hat{x} - (1 - d)\hat{x})] \\ &\quad + a(\hat{x} - (1 - d)\hat{x}) - a(g(x) - (1 - d)x) + \rho[W(\hat{x}) + a(1 - d)(g(x) - \hat{x})] \\ &= W(\hat{x}) + a(\hat{x} - (1 - d)\hat{x}) - a(g(x) - (1 - d)x) + \rho a(1 - d)(g(x) - \hat{x}) \\ &= W(\hat{x}) - [1 - \rho a\xi(1 - d)](\hat{x} - x) = W(x) \end{aligned}$$

For $(x, x) \in \Omega$ with $x' \in Y$ and $x' > g(x)$, then:

$$\begin{aligned} u(x, x') + \rho W(x') &\leq [1 - a(x' - (1 - d)x)] + \rho W(x') \\ &= [1 - a(g(x) - (1 - d)x)] + a(g(x) - x') \\ &\quad + \rho[W(\hat{x}) + a(1 - d)(x' - \hat{x})] = [1 - a(g(x) - (1 - d)x)] + a(g(x) - x') \\ &\quad + \rho[W(\hat{x}) + a(1 - d)(g(x) - \hat{x})] + \rho a(1 - d)(x' - g(x)) \\ &= W(x) + a(x' - g(x))[\rho(1 - d) - 1] < W(x) \end{aligned}$$

For $(x, x') \in \Omega$ with $x' \in Y$ and $x' < g(x)$, then $u(x, x') \leq x = u(x, g(x))$ and $W(x') < W(g(x))$, so that:

$$u(x, x') + \rho W(x') < u(x, g(x) + \rho W(g(x)) = W(x)$$

This completes our verification that equation (60) and (61) hold for every $x \in Y'$. □

We can now use Lemma 7 to establish the optimality of the pan policy when $\rho\xi > 1$, and we state this result separately as a proposition.

PROPOSITION 4. *Suppose the parameters of the model satisfy the condition:*

$$\rho\xi > 1 \tag{63}$$

Then, for every $x \in (\hat{x}, k)$, we have:

$$\hat{x} \in h(x)$$

Proof. Suppose, on the contrary, that there is some $x_o \in (\hat{x}, k)$ such that $\hat{x} \notin h(x_o)$. Consider an optimal program $(\bar{x}(t), \bar{y}(t))$ from x_o, and define a sequence $(x(t))$ by $x(0) = x_o$, and

$x(t+1) = \hat{x}$ for $t \geq 0$. Then, $(x(t), x(t+1)) \in \Omega$ for $t \geq 0$, and defining $y(t) = \Lambda(x(t), x(t+1))$ for $t \geq 0$, we know that $(x(t), y(t))$ is a program from x_o. Since $\hat{x} \notin h(x_o)$, we must have:

$$\sum_{t=0}^{\infty} \rho^t u(x(t), x(t+1)) < \sum_{t=0}^{\infty} \rho^t u(\bar{x}(t), \bar{x}(t+1)) \tag{64}$$

For $\lambda \in (0, 1)$, define $\tilde{x}(t) = \lambda x(t) + (1-\lambda)\bar{x}(t)$ and $\tilde{y}(t) = \lambda y(t) + (1-\lambda)\bar{y}(t)$ for $t \geq 0$, then, by convexity of Ω, $(\tilde{x}(t), \tilde{y}(t))$ is a program from x_o, and:

$$\sum_{t=0}^{\infty} \rho^t u(x(t), x(t+1)) < \sum_{t=0}^{\infty} \rho^t u(\tilde{x}(t), \tilde{x}(t+1)) \tag{65}$$

By choosing λ sufficiently close to 1, one can ensure that $\tilde{x}(t) \in Y = [K, k]$ for all $t \geq 0$, since $x(t) = \hat{x}$ for $t \geq 1$, $\bar{x}(t) \in Z \equiv [0, 1/ad]$ for $t \geq 1$, and $\hat{x}$ is in the interior of Y. But, then, we can apply Lemma 7 to the program $(\tilde{x}(t), \tilde{y}(t))$ (for such λ), to obtain:

$$W(\tilde{x}(t)) \geq u(\tilde{x}(t), \tilde{x}(t+1)) + \rho W(\tilde{x}(t+1)) \quad \text{for } t \geq 0$$

so that:

$$W(x_o) \geq \sum_{t=0}^{\infty} \rho^t u(\tilde{x}(t), \tilde{x}(t+1)) \tag{66}$$

Since $x(t+1) = f(x(t))$ for $t \geq 0$, we can also use Lemma 7 to obtain:

$$W(x(t)) = u(x(t), x(t+1)) + \rho W(x(t+1)) \quad \text{for } t \geq 0$$

so that:

$$W(x_o) = \sum_{t=0}^{\infty} \rho^t u(x(t), x(t+1)) \tag{67}$$

Clearly, equation (66) and (67) together contradict equation (65), establishing the proposition. □

Remarks.

(i) It is clear from Lemma 7 that the conclusion in Proposition 4 and in part (i) of Theorem 1 can be strengthened to read $\hat{x} = h(x)$ for all $x \in (\hat{x}, k)$, when $\rho\xi > 1$; that is, the pan policy, defined by g, is the optimal policy function when $\rho\xi > 1$.

(ii) The pan policy function, g, defined in equation (21), is precisely the optimal policy function in the RSS model obtained in the undiscounted case [5]), when $\xi > 1$. Thus, Proposition 4 establishes that for $\xi > 1$, the qualitative properties of optimal programs are the same in the discounted case, for all discount factors $(1/\xi) < \rho < 1$, as in the undiscounted case. In particular, any optimal program $\{x(t), y(t)\}$ must satisfy a (straight down the) turnpike property: $x(t) = \hat{x}$ for all t, after at most a finite number of periods.

5. Concluding remarks

The bifurcation result of Theorem 1 naturally raises two related questions. First, what kind of optimal behavior would one observe at the bifurcation value of the discount factor $\rho^* = (1/\xi)$? Second, what is the optimal policy correspondence when $\rho < (1/\xi)$?

It is possible that at this threshhold discount factor the optimal policy correspondence is given precisely by G, described in equation (23). And, then, for $\rho < (1/\xi)$, the graph of the optimal policy correspondence is the lower boundary of the graph of G, which can be referred to as the "check" policy function. In this case, the analogy is complete with the undiscounted case: the bifurcation takes exactly this form, as demonstrated in Ref. [5].

However, a more intricate picture is also possible, in which at the bifurcation value of the discount factor, and indeed for some values of the discount factor lower than this value, only a subset of the transitions (in the middle section), described by the correspondence G, are optimal, and there is a cascade of bifurcation values of the discount factor before one reaches the discount factor at which the "check" policy function is optimal.

Clearly, answering the two questions requires a complete bifurcation analysis of our model with respect to the discount factor. This goes well beyond the modest, but essential, first step undertaken in the current paper, and we hope to report our complete results along these lines in the not too distant future.

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